Differential Equations and Reduction Formulae in Integration

Module – 4

Synopsis:

INTEGRATION READY RECKONER

1. 

2. Indefinite integrals of standard functions : -

|  |  |  |  |
| --- | --- | --- | --- |
| FUNCTION | INTEGRAL | FUNCTION | INTEGRAL |
|  |  | - co | Coth x |
| Cos x | Sin x | - Sech x tanh x | Sech x |
| sin x | - Cos x | - cosech x coth x | Cosech x |
|  | Tan x |  |  |
| co | - Cot x |  |  |
| Sec x tan x | Sec x |  |  |
| cosec x cot x | - Cosec x |  |  |
| 1/x | Log |  |  |
|  |  |  |  |
| log a |  |  |  |
| Cosh x | Sinh x |  |  |
| Sinh x | Cosh x |  |  |
|  | Tanh x |  |  |

Note :- We add the constant of integration c throughout .

3. Rules of Integration : - a) 

b) .

c)  [ Integration by parts ] .

4. Standard Integrals :-

a)  .

b)  .

c) 

d) 

e) 

f) 

g) 

h) 

i) 

j) 

k) 

l) 

m) n) 

o) p) 

q) 

r) 

s) 

t) 

u) 

v) 

w) 

x) 

xi) 

xii) 

Rules to integrate:

i) 

ii) 

iii) 

5. Rules to integrate: 

1. If the index of sin x is a positive odd integer, put cosx = t.
2. If the index of cos x is a positive odd integer, put sin x = t

* Reduction Integration

1. 

2.  

3. 

4.  

5.  

6. 

(i) when n is even and m is even



(ii) when n is even and m is odd



(iii) when n is odd and m is even or odd,



* Properties of Definite Integral

(i) 

(ii) 

(iii) 

(iv) 

(v) 

(vi) 

* To evaluate some definite integrals the following substitution is used.

|  |  |  |
| --- | --- | --- |
| SL.No |  | Substitution |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |
| 7 |  |  |
| 8 |  |  |
| 9 |  |  |
| 10 |  |  |
| 11 |  |  |
| 12 |  |  |
| 13 |  |  |
| 14 |  |  |
| 15 |  |  |

* Ordinary Differential Equations

1. is a Linear differential equation then the solution is  or   
 

2. BERNOULLI’S EQUATION    
 

5. EXACT DIFFERENTIAL EQUATION

 is an exact differential equation. The condition for exactness is

 & the general solution is 

6. REDUCIBLE TO EXACT

* If  then the integrating factor .
* If  then the integrating factor .
* If  are homogeneous and , then the integrating factor .
* If ,  then the integrating factor is .
* If , then the integrating factor is , where h and k to be chosen such that  holds.

7. If  then the two curves are orthogonal.

Module-4 SYLLABUS

Reduction formula, , (*m* and *n* are positive integers.) Evaluation of these integrals with standard limits  and problems.

4.0 Introduction:

In many of the engineering field we come across the integration, if the integral is simple it can be easily integrated. But there are many situations, where evaluation of integration is difficult, in algebraic functions and product of trigonometric functions and its higher powers. By applying the reduction formulae we can easily evaluate the integral.

Any formula which expresses an integral in terms of another integral, which can be integrated easily, is called a reduction formula for the first integral. Reduction formula reduces a given integral to a known integration by the repeated application of integration by parts. Here reduction formulae of only trigonometric functions  are considered i) without limits ii) with standard limits.

4.1 REDUCTION FORMULAE:

Reduction formula for  where n is a positive integer. (VTU 2003, 2004, 2005, 2017J)

Solution: Let  (1)

Rewriting the equation (1) as 

Apply Integration by parts we get









 [ from equation (1)]









Or  (2)

From equation (1) to (2) we observe that there is a reduction in the power from n to n-2. Therefore the equation (2) is the reduction formula for .

4.2 Evaluate .

Solution: Let 

We know that from the reduction formula (2),................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................................





 []

 (3)

This is the Recurrence formula for 

Replace  in (3), we get 

Replace  in (3), we get 

Replace  in (3), we get  and so on.

Preceding in this way, at the end we have two cases.

Case (i) when n is even, Case (ii) when n is odd,

put n = 4 in (3) we get  put n = 5 in (3) we get 

put n = 2 in (3)  put n = 3 in (3) 

and  (4) and  (5)

using the back substitution we get

 (6)

Combining these two results we get 

Using this reduction formula the value of In may be determined.

4.3 Reduction formula for  where n is a positive integer.

Solution: Let  (7)

Rewriting the equation (1) as 

Apply Integration by parts we get









 [ from equation (7)]









Or  (8)

From equation (7) to (8) we observe that there is a reduction in the power from n to n-2. Therefore the equation (8) is the reduction formula for .

4.4 Evaluate . (VTU 2017J)

Solution: Let 

We know that from the reduction formula (8), 





 []

 (9)

This is the Recurrence formula for 

Replace  in (9), we get 

Replace  in (9), we get 

Replace  in (9), we get  and so on.

Proceeding in this way, at the end we have two cases.

Case (i) when n is even, Case (ii) when n is odd,

put n = 4 in (9) we get  put n = 5 in (9) we get 

put n = 2 in (9)  put n = 3 in (9) 

and  (10) and  (11)

using the back substitution we get



Combining these two results we get 

Using this reduction formula the value of *In* may be determined.

4.5 Reduction formula for  where m, n is a positive integer. (VTU 2017J)

Solution: Let 

 



Apply integration by parts we get





















 (12)

4.6 Evaluate 

Solution: we have from Equation (12), 

Let





 (13)

Put n = n – 2 in (13), 

Put n = n – 4 in (13), 

Put n = n – 6 in (13),  and so on.

Proceeding in this way we have two cases

Case (i): n is even number,

Put n = 4 in (13), 

Put n = 2 in (13), 

Where 

[According to (6)]



Case (ii): when n is odd,

Put n = 5 in (13), 

Put n = 3 in (13), 

Where 



Thus (i) when n is even and m is even



(ii) when n is even and m is odd



(iii) when n is odd and m is even or odd,



Problems:

1. Evaluate 

Solution: From the reduction formula  (1)

Put n = 7 in (1), 













2. Evaluate 

Solution: From the reduction formula 

Put n = 8 in (1), 







 

3. Evaluate 

Solution: Here n = 5, odd number, then according to reduction formula,



4. Evaluate 

Solution: Here n = 8 even number 

5. Evaluate 

Solution: put  When 





6. Evaluate 

Solution: put  When 





7. Evaluate 

Solution: Let  









8. Evaluate 

Solution: Let 



 put 

When 



9. Evaluate 

Solution: Let 



 put 

When 



10. Evaluate 

Solution: Let  put 

When 



11. Evaluate  (VTU 2006)

Solution: Let  



11. Evaluate 



Solution: Let  put 

When 







12. Evaluate 

Solution: Let  



Apply Integration by parts and 





 If n is odd  If n is even.

14. Evaluate 

Solution: Let  







15. Evaluate 

Solution: Let 



 



16. Evaluate 

Solution: Let  





17. Show that, for any positive integer n, . Hence evaluate.

Solution: Let 









 



 (1)

Take n = 3 and a = 5, in (1) we get 

18. Evaluate 

Solution: Let 

Using the reduction formula 









19. Evaluate 

Solution: Let 

Using the reduction formula 



















20. Evaluate 

Solution: Let , Here *n = 8* is an even number

Using the reduction formula, 



21. Evaluate 

Solution: Let  Here n = 7 is an odd number

Using the reduction formula, 



22. Evaluate 

Solution: Let 





23. Evaluate 

Solution: Let 



 



24. Evaluate 

Solution: Let  (1)

Using the property 



 [from (1)]





25. Evaluate 

Solution: Let  



26. Evaluate 

Solution: Let 

 

27. Evaluate 

Solution: Let  





 

28. Evaluate  (VTU 2008S)

Solution: Let







 

29. Evaluate  where n is a positive integer. Hence show that .

Solution:  





 

 (1)

Replacing n by n – 1 in (1), we get

 (2)



30. Evaluate 

Solution: we have from the reduction formula, 

Taking m = 4 and n = 2













31. Evaluate 

Solution: 

Using the formula 



32. Evaluate 

Solution: Let  using 





33. Evaluate 

Solution:  using 



34. Evaluate 

Solution: Let 

Using 









35. Evaluate 

Solution: Let 











36. Evaluate 

Solution: Let 



 





37. Evaluate  (VTU 2003S)

Solution: 







38. Evaluate 

Solution: 









39. Evaluate  (VTU 2014)

Solution:  







40. Evaluate  (VTU 2017J)

Solution: 











41. Evaluate  (VTU 2013)

Solution:  







42. If n is a positive integer, show that and hence evaluate

 (VTU 2007, 2004)

Solution: 

















Thus 

Taking a = 2 and n = 3, we get



Exercise:

Evaluate the following integrals:

1.  2.  3.  4.  5.

6.  7.  8.  9. 

10.  11.  12. 13. 

14 15.  16.  17. 

18. 

Answers: 1.  2.  3.  4.  5.  6. 

7.  8.  9.  10.  11.  12. 

13.  14.  15.  16.  17.  18. 

4.7 Differential equations

Syllabus:Solution of first order and first degree differential equations– Exact, reducible to exact and Bernoulli’s differential equations. Orthogonal trajectories in Cartesian and polar form. Simple problems on Newton's law of cooling.

4.7.1 Introduction: Differential equations have a wide range of applications in mechanical and electrical systems. Bending of beams, conduction of heat etc., the study of Differential equation involves three basic phases namely i) formulation or mathematical modeling of the equation, ii) solution of the model and iii) geometrical interpretation of the solution.

In this section we introduce differential equations and mention various standard methods of solving simple differential equations of first-order and first-degree to extend our knowledge about differential equations. The method of determining the orthogonal trajectories through differential equation is also illustrated.

An equation involving with derivative and their variables is called a Differential equation.

Differential equations having only one independent variable and derivatives with respect to it are called Ordinary differential equation.

Differential equations which involve two or more independent variables and partial derivatives with respect to them are called Partial differential equation.

The Order of a differential equation is the highest order derivative present in the equation. The Degree of a differential equation is the power of the highest derivative present in the equation. Order and degree of a differential equation are always positive integers.

Examples:

1. is an ordinary differential equation. The order of the equation is one and degree is two.

2. is an ordinary differential equation of order 2 and degree two.

3. is ordinary differential equation of order two and degree six.

4. is a partial differential equation. Here z is a function of independent variable x and y.

4.7.2 Formation of differential equation

Differentiate the given equation with respect to the variable and eliminate the arbitrary constants. The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Note: As, the differentiation of a constant is zero ; we add a constant in each time while integrating a function. the number of constants added is equal to the number of times integrating the function. Thus, as a converse, to get a deferential equation we have to eliminate the constants from the given equation by differentiating as many times as the number of constants contained in it.

Problems

1. Form the differential equation whose solution is



Solution: Given

 (1)

Differentiating (1) with respect to x, we get



 (from (1)) (2)

Differentiating again with respect to x, we get







 (from (2))



Hence,



2. Form the differential equation of all circles touching the axis of y at the origin and centers on the axis of x.

Solution: Since center of the circle is on x-axis, the center is . Further it touches the y-axis at the origin the radius is *a*. Thus the equation to the required circle is

 (1)

Now, differentiating (1), with respect to x, we get

 (2)

 (3)

Substituting (2) and (3) in (1) we get





3. Form the differential equation of all circles of radius a.

Solution: The equation to the circle of radius *a* is

 (1)

where *h* and *k* are parameters, *(h, k)* is the center of the circle.

Now differentiating (1), with respect to *x*, we get

 (2)

Now differentiating (2), again with respect to *x*, we get

 (3)

Substituting (3) in (1), we get

 (4)

Now substituting (3) in (4), we get



4. Form the differential equation whose solution is 

Solution: given  …. (1)

Differentiating (1), with respect to *x,* we get

 ….. (2)

Substituting (2) in (1), we get



5. Form the differential equation whose solution is



Solution: Given  (1)

Differentiating (1), with respect to *x*, we get

 (2)

Substituting in (1), we get



6. Form the differential equation whose solution is 

Solution: Given  (1)

Differentiating (1), with respect to *x*, we get

 (2)

Differentiating (2), again with respect to *x*,



 (from (1))

4.7.3 Solution of a differential equation

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation. A solution in which the number of arbitrary constants is equal to the order of the differential equation is called General (or complete) solution of the differential equation. A solution obtained by giving particular (definite) value to the arbitrary constants is called Particular solution.

A differential equation may have a single solution or many solutions or no solution.

A differential equation together with a given condition (initial condition) is called initial value problem (IVP)

4.8 Equations of First order and first degree:

We discuss some special methods of solving the following types of equations:

(i) Equations where variables are separable, (ii) Homogeneous equations

(iii) Linear equations, (iv) Exact equations

4.8.1 Variables separable:

If in a differential equation containing two variables say *x* and *y* it is possible to collect the function of *x* with *dx* as a single term leaving only the function *y* and *dy* as another term, then the differential equation is called Variable separable. The variables occurring in the differential equation are then said to be separable. The general form of the differential equation in which the variables be separated is

The solution of the differential equations is



1. Solve  (VTU 2012)

Solution: Given equation is 



Integrating both sides, 







 or 



2. Solve 

Solution: Given equation is 

On dividing by tanx tany, we get

 (variables separated)

Integrating  



Or 

3. Solve  (VTU 2009)

Solution: Given equation is 

Separating of variables 

Integrating

 or 

4. Solve 

Solution: Given equation is 





Separating of variables 



Integrating

  or 

5. Solve 

Solution: Given equation is 

Separating of variables 

Integrating





6. Solve 

Solution: Given equation is 

Separating the variables

Integrating 

 





4.8.2 Homogeneous differential equation

If it is possible to write the given differential equation in the form  (1)

where *f(x, y)* and *g(x, y)* both are homogeneous functions of same degree in x and y then the differential equation is called homogeneous differential equation.

Since both *f(x, y)* and *g(x, y)* are homogeneous functions of the same degree say n in x and y, we have

 (2)

Setting we get y = vx and  hence the equation (2) becomes 

 (3)

Since the right hand side of the equation (3) is a function of v only, (3) can be written as  (4)

which is a variable separable form.

Hence by substituting , equation (1) will be reducible to variable separable.

Note: In some times it is necessary to substitute  to reduce variable separable form that can be further integrable.

Problems:

1. Solve 

Solution: Given differential equation  (1)

is homogeneous.

Put  so that , (1) becomes





 [variable separable]

Integrating both sides we get, 







 or 

2. Solve 

Solution: Given differential equation  (1)

is homogeneous.

putting  so that , (1) becomes





 (variable seperable)



Integrating both sides we get,











3. Solve 

Solution: Given differential equation  (1)

is homogeneous.

putting  so that , (1) becomes





 (variable separated)

Integrating both sides we get,

















4. Solve: 

Solution:  (1)

is homogeneous.

Putting  so that , (1) becomes









 (variable separated)

Integrating both sides,









4.8.3 Linear Equations

A differential equation is said to be Linear differential equation, if the dependent variable and its derivatives occur only in first degree and are not multiplied together.

The general form of a linear differential equation is  (1)

Where p(x) and Q(x) are functions of x alone.

To obtain a solution of (1), multiply (1) by a function  such a function we multiply is called Integrating factor (I.F) then equation (1) becomes 



Integrating both sides with respect to x, we get solution of (1) as





Similarly, For the differential equation  (2)

where p(y) and q(y) are functions of y alone.

The Integrating factor is 

Solution for equation (2) is 



1. Solve 

Solution: Given equation is 

Rewriting the equation as or 

This is of the form  with 

Integrating Factor is 

Therefore the solution is 

 







Or 

2. Solve 

Solution: Given equation is 

Rewriting the equation we get 



This is of the form  with 

The Integrating factor is 

Therefore the solution is 

 





3. Solve 

Solution: Given differential equation is 

Rewriting the equation 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 

 



Or 

4. Solve 

Solution: Given differential equation is 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 

 



Or 

Taking , we get





Therefore the required solution is 

5. Solve 

Solution: Given differential equation is 

Rewriting the equation 



This is of the form  with 

The Integrating factor is 

Therefore the solution is 

 







6. Solve 

Solution: Given differential equation is 

Rewriting the equation 



This is of the form  with 

The Integrating factor is 

Therefore the solution is 





7. Solve: 

Solution: Given equation is 

Rewriting the equation 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 





8. Solve: 

Solution: Given equation is 

Rewriting the equation  (1)

This is of the form  with 

The Integrating factor is 

Therefore the solution is 



 





9. Solve: 

Solution: Given equation is 

Rewriting the equation  or 



This is of the form  with 

The Integrating factor is 

Therefore the solution is 





10. Solve: 

Solution: Given equation is 

Rewriting the equation 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 



Or 

11. Solve: 

Solution: Given equation is 

Rewriting the equation 

Or 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 





 or 

12. Solve: 

Solution: Given equation is 

Rewriting the equation 

Or 

This is of the form  with 

The Integrating factor is 

Therefore the solution is 

 





Exercise:

Solve the following differential equations:

1.  2. 

3.  4. 

5.  6. 

7.  8. 

9.  10. 

11.  12. 

Answers:

1.  2. 

3.  4. 

5.  6. 

7.  8. 

9.  10. 

11.  12. 

4.8.4 BERNOULLI’S EQUATION

The equation  (1)

where P(x) and Q(x) are functions of x, is reducible to the Leibnitz’s linear equation and is usually called the Bernoulli’s equation.

To solve such equation (1), divide both sides by , so that  (2)



 (2) becomes 

Which is a linear differential equation in t and can be solved by the method discussed in the above section.

1. Solve 

Solution: Given equation is 

Dividing throughout by *xy6*,  (1)



 

This is a Linear differential equation in t, with 

The Integrating factor is 

Therefore the solution is 





 is the required solution.

2. Solve  (VTU 2017J)

Solution: Given equation is 

Rewriting the equation or 

Dividing by x2, we have  (1)





This is a Linear differential equation in *t*, with 

The Integrating factor is 

Therefore the solution is 









Or 

2. Solve 

Solution: Given equation is 

Rewriting the equation  OR 





This is a Linear differential equation in *t*, with 

The Integrating factor is 

Therefore the solution is 





Or  is the required solution.

3. Solve  (VTU 2006)

Solution: Given equation is 

Rewriting the given equation 

Or 

 

 or 

This is a Linear differential equation in *t*, with 

The Integrating factor is 

Therefore the solution is 





 is the required solution.

4. Solve  (VTU 2009)

Solution: Given equation is 

Rewriting the equation  

 this is a linear differential equation in t with 

Therefore Integrating factor is 

Therefore the solution is 



Or  is the required solution.

5. Solve  (VTU 2011)

Solution: Given equation is 

Rewriting the equation 

Or  (dividing by )

 

 or 

This is a linear differential equation in t w r to y and 

Therefore Integrating factor is 

Therefore the solution is 





Or  is the required solution.

6. Solve  (VTU 2013)

Solution: Given equation is 

Dividing the equation by 

 



This is a linear differential equation in t with 

Therefore Integrating factor is 

Therefore the solution is 

 or  





 is the required solution.

7. Solve 

Solution: Given equation is 

Rewriting the equation as 

Dividing by y2, we get

 

 or 

This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 



Or  is the required solution.

8. Solve 

Solution: Given equation is 

Rewriting the equation, we get 

 (Dividing by y4)



 or 

This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 



or 

9. Solve 

Solution: Given equation is 

Rewriting the equation, we get  or  (Dividing by y2)

 

 or 

This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 

 







10. Solve 

Solution: Given equation is 

Rewriting the equation or 

Multiply by 2y,

 



This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 





Or  is the required solution.

11. Solve 

Solution: Given equation is 

Rewriting the equation 





This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 





Or  is the required solution.

12. Solve 

Solution: Given equation is 





This is a linear differential equation in t with 

The Integrating factor is 

Therefore the solution is 



 



 or 

Exercise:

1.  2. 

3.  4. 

5.  6. 

7.  8. 

9.  10. 

Answers:

1.  2. 

3.  4.  5. 

6.  7. 

8. 

9. 

4.8.5 EXACT DIFFERENTIAL EQUATIONS

Definition: A differential equation of the form  is said to be exact if its left hand member is the exact differential of some function f(x, y) i.e., . Its solution, therefore is f(x, y) = c.

The necessary and sufficient condition for the differential equation  to be exact is 

The solution of  is 

Problems:

1. Solve  (VTU 2006)

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 is the required solution.

2. Solve  (VTU 2006)

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 is the required solution.

3. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 

 or 

 is the required solution.

4. Solve 

Solution: The given equation is 

Rewriting the equation 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 

 or 

 is the required solution.

5. Solve  (VTU 2008)

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is  



 or 

6. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 or 



7. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 or  is the required solution.

8. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 is the required solution.

9. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 





Or  is the required solution.

10. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 is the required solution.

11. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 is the required solution.

12. Solve 

Solution: The given equation of the type 

Here 

 

Thus,  Hence the given differential equation is exact.

Therefore the solution is 



 or 

Exercise:

Solve the following:

1.  2. 

3.  4. 

Answers:

1.  2. 

3.  4. 

4.8.6 REDUCIBLE TO EXACT DIFFERENTIAL EQUATION

The equation  can be reducible to exact differential equation by multiplying throughout the equation by the integrating factor. The suitable integrating factors are given below:

* If  then the integrating factor .
* If  then the integrating factor .
* If  are homogeneous and , then the integrating factor .
* If ,  then the integrating factor is .
* If , then the integrating factor is , where h and k to be chosen such that  holds.

Problems:

1. Solve 

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Since the given equation is homogeneous in x and y.

And 

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 



 is the required solution.

2. Solve  (VTU 2014)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.



And 

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 





 is the required solution.

3. Solve  (VTU 2009)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

and  a function of x alone.

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 



 



 is the required solution.

4. Solve  (VTU 2013)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

or  is a function of y alone.

Therefore 

Multiply (1), by  we get 

 (2)

This is of the type M dx + N dy = 0 with 

 

Thus,  Hence the given differential equation is an exact.

Therefore the solution is 



 or  is the required solution.

5. Solve 

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

or  is a function of y alone.

Therefore 

Multiply (1), by  we get  (2)

This is of the type M dx + N dy = 0 with 

 

Thus,  Hence the given differential equation is an exact.

Therefore the solution is 

 



 is the required solution.

6. Solve 

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also the equation (1) can be rewritten as  (2)

This is of the form 

With 

Therefore the integrating factor is 

Multiply the equation (1) by  we get



 (3)

Here 

 

Since equation (3), is exact we have 



Equating the coefficient of  we get

 or 

Solving the above two equation we get 



Or 

Therefore the solution is 



 or  is the required solution.

6. Solve  (VTU 2009)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.



And 



Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 





 is the required solution.

7. Solve  (VTU 2009)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.



And 



Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 





 or  is the required solution.

8. Solve 

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

or  is a function of y alone.

Therefore 

Multiply (1), by  we get 

 (2)

This is of the type M dx + N dy = 0 with 

 

Thus,  Hence the given differential equation is an exact.

Therefore the solution is 



 is the required solution.

9. Solve 

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

and  a function of x alone.

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 





 is the required solution.

10. Solve  (VTU 2017J)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

and  a function of x alone.

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 





 is the required solution.

11. Solve  (VTU 2017J)

Solution: The equation is of the type

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Also 

and  a function of x alone.

Therefore 

Multiply (1), by  we get  (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 



 is the required solution.

12. Solve 

Solution: The given equation is 

 (1)

Here

 

Thus,  Hence the given differential equation is not an exact.

Since the given equation is homogeneous in x and y.

And 

Therefore 

Multiply (1), by  we get 

 (2)

Here

 

Thus,  Hence the differential equation (2) is exact.

Therefore the solution is 



 is the required solution.

Exercises:

Solve the following differential equations:

1.  2. 

3.  4. 

5.  6. 

7.  8. 

9.  10. 

11.  12. 

13.  14. 

15. 

Answers:

1.  2. 

3.  4.  5. 

6.  7.  8. 

9.  10.  11. 

12.  13.  14. 

15. 

4.9 ORTHOGONAL TRAJECTORIES:

In this section for a given set of curves we construct a new family of curves that cuts every member of the set perpendicularly. These curves are very essential in electrical engineering and other allied branches.

Consider a family of curves. A curve, which cuts every member of the other family of curves, is called trajectory of the family.

A curve, which cuts every member of the other family orthogonally, is called orthogonal trajectory of the family.

Two families of curves are said to be orthogonal trajectories of each other if each member of one family cuts every member of the other family orthogonally.

Example : consider a set 

of family of all possible circles centered at the origin and a

set  of all straight lines passing

through the origin. Then each line in F2 cuts every circle in

F1 orthogonally, and vice versa. Hence the sets F1 and F2 are

orthogonal trajectories of each other.

4.9.1 Orthogonal trajectory of Cartesian curves:

Since two Cartesian curves are orthogonal if the product of slopes of their tangents is -1 at the point of intersection, the slope of the orthogonal curve is the reciprocal of the negative value of the slope of the curve. More precisely if  is the slope of the given curve, then the slope of the curve orthogonal to it is . Thus we have the following working rule to find orthogonal trajectories of the curves given in Cartesian format

Working rule to find orthogonal trajectory in Cartesian form:

* Obtain a differential equation of the family of curves by eliminating the parameters present in its equation by differentiating with respect to x or y whichever is convenient.
* Replace  in the differential equation obtained in above step.
* Solve the resultant differential equation obtained in 2nd step.

4.9.2 Orthogonal trajectory of polar curves

We have two polar curves are orthogonal if , that is at the point of intersection (at this point r as well as  is same for both the curves)

. Thus, akin to Cartesian curves we have the following working rule,

Working rule to find or1thogonal trajectory in polar form:

* Obtain a differential equation of the family of curves by eliminating the parameters present in its equation by differentiating with respect to  whichever is convenient.
* Replace  in the differential equation obtained in first step
* Solve the resultant differential equation obtained second step.

A curve is said to be Self-orthogonal if it intersect orthogonally. Thus for such curves the resultant differential equation obtained in second step is coincides with the one obtained in first step.

Problems:

1. Find the orthogonal trajectory of the family of parabolas .

Solution: Given curve is  taking log on both sides, we get

 (1)

Differentiating (1), with respect to x, we get  (2)

Equation (2) is the required differential equation containing no parameter.

Replacing  in (2), we get



   or is the orthogonal trajectory.

2. Find the orthogonal trajectory of the family of hyperbola  which passes through the point (1,1).

Solution: Given curve is  (1)

Differentiating (1), with respect to x, we get

 (2)

Equation (2) is the required differential equation containing no parameter.

Replacing  in (2), we get 

Integrating we get the solution  (3)

This is the orthogonal trajectory for the given curve. Now by substituting the passing point x = 1, y = 1 in (3) we get k = 0, thus the orthogonal trajectory passing through (1, 1) is   this is a straight line.

3. Find the orthogonal trajectory of family of coaxial circles 

Solution: Given curve is  (1)

Differentiating (1), with respect to x, we get

 (2)

Multiplying equation (2) by x, we get  (3)

Now subtracting equation (1) from (3), we get  (4)

Equation (4) is the required differential equation containing no parameter.

Replacing  in (4), we get 

 (5)

Equation (5) is of the form,  with 



 hence the equation is not exact.

But, 

Therefore, the integrating factor 

Now multiplying (5) by  we get

 which is an exact differential equation.

The solution of this is 



 is the required orthogonal trajectory.

4. Find the orthogonal trajectory of the family of confocal ellipses 

Solution: Given curve is  (1)

Differentiating (1), with respect to x, we get



Dividing throughout by 2 and multiplying by y we get

 (2)

From (1), we have , substituting this in (2),

we get  (3)

equation (3) is the required differential equation containing no parameter.

Replacing  in (3), we get 

 (4)

Solving the equation (4),

Dividing the equation (4) by x, the variables are seperable 

Integrating 

 is the required orthogonal trajectory.

5. Prove that the system of confocal and coaxial parabolas  is self-orthogonal.

Solution: Given curve is  (1)

Differentiating (1), with respect to x, we get   (2)

Substituting (2) in (1), we get 

 (3)

Equation (3) is the required differential equation containing no parameter.

Replacing  in (3), we get

 (4)

Since the differential equations (4) and (3) are identical, the given curve is self-orthogonal.

6. Prove that the system of confocal conics , is self-reciprocal.

Solution: Given curve is  (1)

Differentiating (1), with respect to x, we get 

 (2)



 (3)

Hence, 





Substituting these in (1), we get 

 (4)

Equation (4) is the required differential equation containing no parameter.

Replacing  in (4), we get



 (5)

Since the differential equation (4) and (5) are identical, the given curve is self orthogonal.

7. Find the orthogonal trajectory of the cardioid 

Solution: Given curve is  (1)

Taking log on both sides 

Differentiating with respect to  we get

 (2)

Equation (2) is the required differential equation containing no parameters.

Replacing  by  in (2)

 (3)

Variables in differential equation (3) are separable, hence the solution is

 



 where 

8. Find the orthogonal trajectory of the curve  (VTU 2017J)

Solution: Given curve is  (1)

Taking log on both sides3



Differentiating with respect to  we get 

 (2)

Equation (2) is the required differential equation containing no parameter.

Replacing  by  in (2)

 (3)

Variables in equation (3) are separable, hence the solution is



 where 

Exercise:

Find the orthogonal trajectories of the following family of curves, where ‘*a* ‘ is the parameter

(i)  (ii)  (iii)  (iv) 

(v)  (vi) 

Answer:

(i)  (ii)  (iii)  (iv)

(v)  (vi) 

4.10 Newton’s Law of Cooling:

The rate of change of the temperature of a body is proportional to the difference between the temperature of the body and that of the surrounding.

Consider

Initial temp of body is T1 0C, is allowed to cool in air medium which is at T2 0C. Let the temperature after time ( t ) be T 0C, Newton’s law is given by



( k ) is the constant of proportionality. Initial condition is given by T = T1 at t = 0, Solving the above equation we get, 

Initially we have T(0) = T1. Simplifying the above equation 

Problems

(1) Suppose the temperature of metal is 3000C when it leaves the oven and 10 min later the temperature is 2000C. When will it be equal to the room temperature of 600C?

Using Newton’s law of cooling, we get





Hence the solution is



Again at t = c, T = 3000C. So we have



When t =10, T = 2000C. So



Hence the solution is T = 60+240e-0.054t. Let at time t = t1, the temperature be 610C



Hence the time required is nearly about 101.5 minutes.

(2) Suppose that an object is heated to 3000C and allowed to cooling a room whose air temperature is 800C. After 10 min the temperature of the object is 2500C. What will be its temperature after 20 min? (VTU 2017J)

Solution

It is given that the temperature of an object is 3000C. After 10 min , temperature of the object decreases to 2500C

Consider temperature as T at time (t)

From Newton’s law we have



Given T0=800C



Separating Variables and Integrating



When t = 0, T = 3000C



Substituting the value of K



When t = 0, T = 250oC



Value of T at t = 20





Exercise

1. A metal ball is heated to a temperature of 100oC and at a time t = 0 it is placed in water which is maintained at 40oC. If the temperature of the ball is reduced to 60oC in 4 min, find the time at which the temperature of the ball is 50oC. ANS: t = 6.5min

2. The temperature of a cup of coffee is 92oC, when freshly poured the room temperature being 24oC. In one minute it was cooled to 80oC. How long a period must elapse, before the temperature of the cup becomes 65oC? ANS: t = 0.58 minutes

3. A body is originally at 80oC cools down to 60oC in 20 minutes, the temperature of the air being 40oC. What will be the temperature of the body after 40 minutes from the original? (VTU 2017J)

4. If the air is maintained at 30oC and the temperature of the body cools from 80oC to 60oc in 12 minutes, find the temperature of the body after 24 minutes.